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An exact solution of the Einstein–Dirac equations

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Abstract. An exact solution to the Einstein–Dirac equations is obtained for a space–time with metric of the anisotropic cosmological type, the gravitational field having as its source a (massive) Dirac electron field.

1. Introduction

To date there are very few exact, 'non-ghost' solutions to the Einstein–Dirac equations in which the Dirac field possesses rest mass—a ghost solution is one for which the energy–momentum tensor of the Dirac field vanishes identically. Most (if not all) of the known solutions are 'ghost' solutions or solutions for a neutrino field with rest mass (see for example Krori *et al* 1982). The solution presented here, although very simple in form, does represent a 'non-ghost' massive electron field.

2. Derivation of the equations and their solution

We assume the metric to have the form

$$ds^2 = dt^2 - e^{2P_1} dx^2 - e^{2P_2} dy^2 - e^{2P_3} dz^2 \tag{1}$$

where the P_i are functions of t only. We assume also that the Dirac current vector, j^α, and energy-momentum tensor, $T^{\alpha\beta}$, are both invariant under the isometries of (1), i.e. that the Lie derivatives of both j^α and $T^{\alpha l}$ with respect to the three vectors $X_1 = \partial/\partial x, X_2 = \partial/\partial y$ and $X_3 = \partial/\partial z$ must vanish. This implies (see Henneaux 1980) that

$$L_{X_j}\psi = ik_j\psi \qquad (j = 1, 2, 3) \tag{2}$$

where L is the Lie derivative,

$$\psi = \begin{pmatrix} u_A \\ \bar{v}^{\dot{B}} \end{pmatrix}$$

is the Dirac bispinor (see RK) and the k_i are constants.

To carry forward our calculation we now introduce a Newman–Penrose (NP) tetrad (see RK)

$$(l_\alpha)=\frac{1}{\sqrt{2}}(1,-e^{P_1},0,0),\qquad (n_\alpha)=\frac{1}{\sqrt{2}}(1,e^{P_1},0,0),$$

$$(m_\alpha)=\frac{1}{\sqrt{2}}(0,0,-e^{P_2},-i\,e^{P_3}).\tag{3}$$

The non-zero spin-coefficients for (3) are

$$\rho=-\mu=-\frac{1}{2\sqrt{2}}(\dot P_2+\dot P_3),\qquad \sigma=-\lambda=\frac{1}{2\sqrt{2}}(\dot P_3-\dot P_2),\qquad \varepsilon=-\gamma=\frac{1}{2\sqrt{2}}\dot P_1,\tag{4}$$

where $\dot P_i=\mathrm{d}P_i/\mathrm{d}t$.

The NP equations then give

$$\begin{aligned}&\phi_{01}=\phi_{12}=0,\qquad \phi_{02}=\phi_{20},\qquad \phi_{00}=\phi_{22},\\ &\ddot P_1+\dot P_1^2-\dot P_2\dot P_3=-4\phi_{11},\qquad \ddot P_2+\dot P_2^2-\dot P_1\dot P_3=2(\phi_{02}-\phi_{00}),\\ &\ddot P_3+\dot P_3^2-\dot P_1\dot P_2=-2(\phi_{02}+\phi_{00}),\qquad \dot P_1\dot P_2+\dot P_1\dot P_3+\dot P_2\dot P_3=2(\phi_{11}+\phi_{00}+3\Lambda).\end{aligned}\tag{5}$$

Now (2) implies that dyad components of the bispinor (u_0, u_1, v_0, v_1) must take the form

$$u_p=\exp[i(k_1x+k_2y+k_3z)]X_p,\qquad v_p=\exp[-i(k_1x+k_2y+k_3z)]Y_p,\tag{6}$$

where $p=0, 1$, the X_p and Y_p being functions of t alone.

Using (6) we can (via the Einstein equations) calculate the ϕ_{pq} (see RK):

$$\begin{aligned}\phi_{00}=\sqrt{2}\,k\,e^{-P}[&4k_1e^{-P_1}(U_0\bar U_0-V_0\bar V_0)+(k_2e^{-P_2}-ik_3e^{-P_3})(U_0\bar U_1-V_0\bar V_1)\\&+(k_2e^{-P_2}+ik_3e^{-P_3})(\bar U_0U_1-\bar V_0V_1)\\&+(1/L)(V_0U_1-U_0V_1+\bar V_0\bar U_1-\bar U_0\bar V_1)],\\
\phi_{22}=\sqrt{2}\,k\,e^{-P}[&-4k_1e^{-P_1}(U_1\bar U_1-V_1\bar V_1)+(k_2e^{-P_2}-ik_3e^{-P_3})(U_0\bar U_1-V_0\bar V_1)\\&+(k_2e^{-P_2}+ik_3e^{-P_3})(\bar U_0U_1-\bar V_0V_1)\\&+(1/L)(V_0U_1-U_0V_1+\bar V_0\bar U_1-\bar U_0\bar V_1)],\\
\phi_{01}=(ke^{-P}/\sqrt{2})[&2k_1e^{-P_1}(U_0\bar U_1-V_0\bar V_1)+3(k_2e^{-P_2}+ik_3e^{-P_3})(U_0\bar U_0-V_0\bar V_0)\\&+(k_2e^{-P_2}+ik_3e^{-P_3})(U_1\bar U_1-V_1\bar V_1)+\tfrac{1}{2}i(2\dot P_1-\dot P_2-\dot P_3)(U_0\bar U_1-V_0\bar V_1)\\&+\tfrac{1}{2}i(\dot P_2-\dot P_3)(\bar U_0U_1-\bar V_0V_1)],\\
\phi_{12}=(ke^{-P}/\sqrt{2})[&2k_1e^{-P_1}(U_0\bar U_1-V_0\bar V_1)+3(k_2e^{-P_2}+ik_3e^{-P_3})(U_1\bar U_1-V_1\bar V_1)\\&+(k_2e^{-P_2}+ik_3e^{-P_3})(U_0\bar U_0-V_0\bar V_0)-\tfrac{1}{2}i(2\dot P_1-\dot P_2-\dot P_3)(U_0\bar U_1-V_0\bar V_1)\\&-\tfrac{1}{2}i(\dot P_2-\dot P_3)(\bar U_0U_1-\bar V_0V_1)],\\
\phi_{02}=\sqrt{2}k\,e^{-P}\{&2(k_2e^{-P_2}+ik_3e^{-P_3})(U_0\bar U_1-V_0\bar V_1)\\&+\tfrac{1}{2}i(\dot P_3-\dot P_2)[(U_0\bar U_0-V_0\bar V_0)-(U_1\bar U_1-V_1\bar V_1)]\},\end{aligned}\tag{7}$$

$$\phi_{11} = (k\,e^{-P}/\sqrt{2})[2(k_2\,e^{-P_2} + ik_3\,e^{-P_3})(\bar{U}_0 U_1 - \bar{V}_0 V_1)$$
$$+2(k_2\,e^{-P_2} - ik_3\,e^{-P_3})(U_0\bar{U}_1 - V_0\bar{V}_1)$$
$$+(1/L)(V_0U_1 - U_0V_1 + \bar{V}_0\bar{U}_1 - \bar{U}_0\bar{V}_1)],$$
$$\Lambda = (k\,e^{-P}/3\sqrt{2}L)(V_0U_1 - U_0V_1 + \bar{V}_0\bar{U}_1 - \bar{U}_0\bar{V}_1),$$

where the Einstein equations are written as $G_{\alpha\beta} = -8kT_{\alpha\beta}$, $P = P_1 + P_2 + P_3$, $X_p = e^{-P/2}U_p$, and $Y_p = e^{-P/2}V_p$.

The Dirac equations take the form (see RK)

$$\begin{aligned}
&\dot{U}_0 - ik_1\,e^{-P_1}U_0 + (-ik_2\,e^{-P_2} + k_3\,e^{-P_3})U_1 = -(i/L)\bar{V}_1,\\
&\dot{U}_1 + ik_1\,e^{-P_1}U_1 - (ik_2\,e^{-P_2} + k_3\,e^{-P_3})U_0 = (i/L)V_0,\\
&\dot{V}_0 - ik_1\,e^{-P_1}V_0 + (-ik_2\,e^{-P_2} + k_3\,e^{-P_3})V_1 = -(i/L)\bar{U}_1,\\
&\dot{V}_1 + ik_1\,e^{-P_1}V_1 - (ik_2\,e^{-P_2} + k_3\,e^{-P_3})V_0 = (i/L)\bar{U}_0.
\end{aligned} \tag{8}$$

If we take the k_i to be non-zero we find, using (5), (7) and (8), that U_p is proportional to V_p (so the field is a massive neutrino field, or type II field of RK) and that the ϕ_{pq} and Λ vanish, making the field a 'ghost' field. In the following we take $k_i = 0$, $i = 1, 2, 3$. The Dirac equations, (8), are now easily solved to give

$$\begin{aligned}
&U_0 = a_0\,e^{it/L} + b_0\,e^{-it/L}, \qquad U_1 = a_1 e^{it/L} + b_1\,e^{-it/L}\\
&V_0 = -\bar{b}_1\,e^{it/L} + \bar{a}_1\,e^{-it/L}, \qquad V_1 = \bar{b}_0\,e^{it/L} - \bar{a}_0\,e^{-it/L},
\end{aligned} \tag{9}$$

where the a's and b's are complex constants.

The Einstein equations are

$$\begin{aligned}
&\ddot{P}_1 + \dot{P}_1^2 - \dot{P}_2\dot{P}_3 = -c\,e^{-P}, \qquad \ddot{P}_2 + \dot{P}_2^2 - \dot{P}_1\dot{P}_3 = -c\,e^{-P},\\
&\ddot{P}_3 + \dot{P}_3^2 - \dot{P}_2\dot{P}_1 = -c\,e^{-P}, \qquad \dot{P}_1\dot{P}_2 + \dot{P}_1\dot{P}_3 + \dot{P}_2\dot{P}_3 = 2c\,e^{-P},
\end{aligned} \tag{10}$$

where $c = (4\sqrt{2}k/L)(a_0\bar{a}_0 + a_1\bar{a}_1 - b_0\bar{b}_0 - b_1\bar{b}_1)$—for a 'non-ghost' solution we require $c \neq 0$.

Equations (10) are now simply solved (for non-trivial, 'non-ghost' solutions) to give the following two solutions (the freedom to rescale x, y, z has been used to eliminate three integration constants):

$$\exp(P_i) = (\tfrac{3}{2}|c|)^{1/3}t^{2/3}, \qquad i = 1, 2, 3, \tag{11}$$

$$P_i = \tfrac{1}{3}P + \frac{\gamma_i}{[3(\gamma_2^2 + \gamma_3^2 + \gamma_2\gamma_3)]^{1/2}} \ln\left(\frac{t - (\gamma_2^2 + \gamma_3^2 + \gamma_2\gamma_3)^{1/2}}{t + (\gamma_2^2 + \gamma_3^2 + \gamma_2\gamma_3)^{1/2}}\right),$$
$$e^P = \exp(P_1 + P_2 + P_3) = \tfrac{3}{2}c[t^2 - (\gamma_2^2 + \gamma_3^2 + \gamma_2\gamma_3)], \tag{12}$$

where $i = 1, 2, 3$ and $\gamma_1 + \gamma_2 + \gamma_3 = 0$.

In either case the Dirac field takes the form

$$\psi = \begin{pmatrix} u_A \\ \bar{v}^B \end{pmatrix} = e^{-P/2} \begin{pmatrix} a_0\,e^{it/L} + b_0\,e^{-it/L} \\ a_1\,e^{it/L} + b_1\,e^{-it/L} \\ -a_0\,e^{it/L} + b_0\,e^{-it/L} \\ -a_1\,e^{it/L} + b_1\,e^{-it/L} \end{pmatrix}$$

with

$$c = (4\sqrt{2}k/L)(a_0\bar{a}_0 + a_1\bar{a}_1 - b_0\bar{b}_0 - b_1\bar{b}_1 \neq 0.$$

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